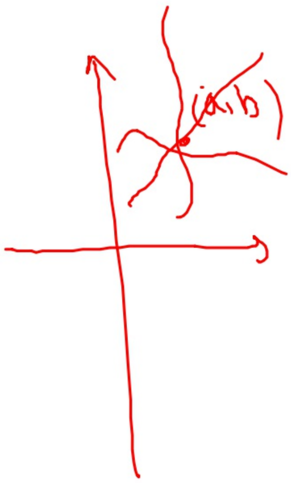
§ 14.2

$$\lim_{x \to a} f(x)$$

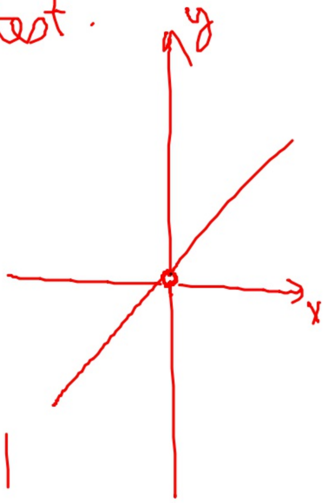
$$\lim_{(x,y) \to (a,b)} f(x,y)$$



two path test:

ex

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2}$$ does not exist by the two path test.



$$\lim_{\substack{x=0 \\ y\to 0}} \frac{2xy}{x^2+y^2} = \lim_{y\to 0} \frac{0}{y^2} = 0$$

y-axis

$$\lim_{\substack{x\to 0 \\ y=x}} \frac{2xy}{x^2+y^2} = \lim_{x\to 0} \frac{2x^2}{2x^2} = 1$$

f cont at (a,b)

if

$$f(a,b) = \lim_{(x,y) \to (a,b)} f(x,y)$$

Remark: $\lim_{(x,y) \to (0,0)} f(x,y) = \lim_{r \to 0} f(r,\theta)$ $\boxed{r \geq 0}$

examples

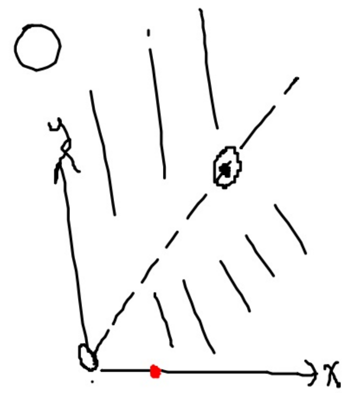
① $$\lim_{(x,y)\to(1,0)} \frac{2x^2+y}{x^2+y^2+2xy^2} = 2.$$

② $$\lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}} = \lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}} \frac{(\sqrt{x}+\sqrt{y})}{\sqrt{x}+\sqrt{y}}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x\cancel{(x-y)}}{\cancel{(x-y)}}(\sqrt{x}+\sqrt{y}) = 0$$

see that $y=x$ is not in Dom(f).

D=



$$D = \{(x,y)\in\mathbb{R}^2 : x\geq 0 \text{ and } y\geq 0 \text{ and } y\neq x\}$$

③ $\lim_{(x,y)\to(0,0)} f(x,y) = \frac{4xy^2}{x^2+y^2} = \lim_{r\to 0} \frac{4r^3\cos\theta\sin^2\theta}{r^2}$

$= \lim_{r\to 0} 4r\cos\theta\sin^2\theta = 0$?? by sand. th.

$-1 \le \cos\theta \le 1$

$-1 \le \cos\theta\sin^2\theta \le 1$

$-4r \le 4r\cos\theta\sin^2\theta \le 4r$

0

Can we extend f to be cont. at $(0,0)$?

Yes!

Let $f(0,0)=0$

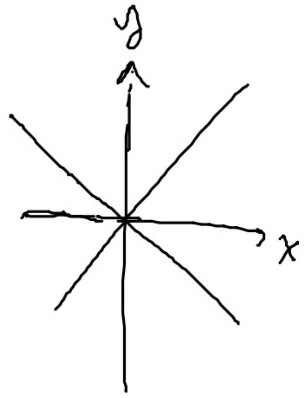
$(0,0)$ is a removable discontinuity.

ex 4

$$f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

See that $Dom(f) = \mathbb{R}^2$

Is f cont at $(0,0)$??



see that $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ y=x}} f(x,y) = \lim_{x\to 0} \frac{2x^2}{2x^2} = 1$

& $\displaystyle\lim_{\substack{y=-x\\ x\to 0}} f(x,y) = \lim_{x\to 0} \frac{-2x^2}{2x^2} = -1$

$\Rightarrow \displaystyle\lim_{(x,y)\to(0,0)} f(x,y) \nexists$

$\therefore$ f is not cont at $(0,0)$.

20

$$\lim_{\substack{(x,y)\to(4,3)\\ x\neq y+1}} \frac{\sqrt{x}-\sqrt{y+1}}{x-y-1} = \lim_{(x,y)\to(4,3)} \frac{\sqrt{x}-\sqrt{y+1}}{x-(y+1)} \times \frac{\sqrt{x}+\sqrt{y+1}}{\sqrt{x}+\sqrt{y+1}}$$

$$= \lim \frac{\cancel{x-(y+1)}}{\cancel{x-(y+1)}} \cdot \frac{1}{\sqrt{x}+\sqrt{y+1}}$$

$$= \frac{1}{4}$$

33) (a) $g(x,y) = \sin \frac{1}{xy}$

g is cont except on the axes in the xy-plane

(b) $g(x,y) = \frac{x+y}{2+\cos x}$ is cont everywhere.

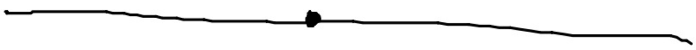
37-b $f(x,y,z) = \frac{1}{x^2+z^2-1}$ is cont in $\mathbb{R}^3$ — the cylinder $x^2+z^2=1$

(4) $f(x,y) = -\frac{x}{\sqrt{x^2+y^2}}$

$$\lim_{\substack{y=0\\ x\to 0}} f(x,y) \;\frac{-x}{\sqrt{x^2+y^2}} = \lim_{x\to 0} \frac{-x}{|x|} = \begin{cases} 1 & \text{if } x<0 \\ -1 & \text{if } x>0 \end{cases}$$

x-axis

$\therefore \lim_{(x,y)\to(0,0)} f(x,y) \nexists$ by the 2-path test



4.3

$$f(x,y)=\frac{x^4-y^2}{x^4+y^2}$$

$$\left.\begin{aligned}
&\lim_{\substack{(x,y)\to(0,0)\\ \text{on}\\ \text{the y-axis}}} f(x,y)=\lim_{\substack{y\to 0\\ x=0}}\frac{-y^2}{y^2}=-1\\[2ex]
&\lim_{\substack{y=0\\ x\to 0}} f(x,y)=\lim_{x\to 0}\frac{x^4}{x^4}=1
\end{aligned}\right\}\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y)\ \nexists$$

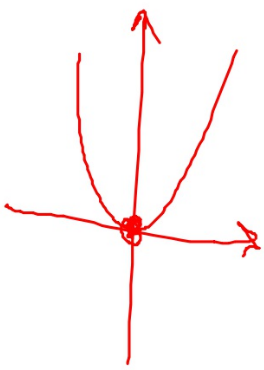
by the 2-path test.

47 $h(x,y) = \dfrac{x^2+y}{y}$

$\lim\limits_{\substack{x=0 \\ y\to 0}} h(x,y) = 1$

y-axis

$\lim\limits_{\substack{y=x^2 \\ x\to 0}} h(x,y) = 2$



$\therefore \lim\limits_{(x,y)\to(0,0)} h(x,y) \nexists$

by the 2-path test.

61 $f(x,y) = \dfrac{x^3 - xy^2}{x^2+y^2}$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} \frac{r^3(\cos^3\theta - \cos\theta\sin^2\theta)}{r^2}$$

$$= \lim_{r\to 0} r\cos\theta(\cos^2\theta - \sin^2\theta)$$

$$= \lim_{r\to 0} r\cos\theta\cos 2\theta = 0 \quad ?? \quad (\text{Sand. th.})$$

$$-1 \le \cos\theta\cos 2\theta \le 1$$

$$-r \le r\cos\theta\cos 2\theta \le r$$

$$\to 0$$

67 $f(x,y) = \ln\left(\dfrac{3x^2 - x^2y^2 + 3y^2}{x^2+y^2}\right)$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} \ln \frac{3r^2 - r^4 \sin^2\theta \cos^2\theta}{r^2}$$

$$= \lim_{r\to 0} \ln\left(3 - r^2 \sin^2\theta \cos^2\theta\right) = \ln 3 \quad ??$$

Let $f(0,0) = \ln 3$

Now f is cont at $(0,0)$

$0 \le \sin^2\theta \cos^2\theta \le 1$

$0 \le r^2 \sin^2\theta \cos^2\theta \le r^2$

0

QUIZ